INDECOMPOSABLE REPRESENTATIONS OF GROUPS WITH A CYCLIC SYLOW SUBGROUP

BY
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1. **Introduction.** D. Higman proved in [6] that the group algebra $K(G)$ of a finite group G over an algebraically closed field K with characteristic p has only a finite number of inequivalent indecomposable modules if and only if the p-Sylow subgroups of G are cyclic. Higman also gave an estimate of the number of such modules. In [7] Kasch, Kneser, and Kupisch gave a sharper upper bound and conditions on G under which the bound would be attained. In this paper we apply J. Green's notion of a vertex and source to give a formula for the exact number of indecomposable $K(G)$-modules.

In §3 we consider the special case $G=LF(2,p)$ and K of characteristic p. Starting with maximal submodules of the principal indecomposable $K(G)$-modules we construct all the indecomposable $K(G)$-modules. The idea is to form certain subdirect sums of these modules. In the Appendix we consider the subdirect sum of two modules over an arbitrary ring, and give a test to determine when the subdirect sum is indecomposable.

2. **The counting formula.** Let G be a finite group and K an algebraically closed field of characteristic p. Recall that a p-subgroup Q of G is called a vertex for an indecomposable $K(G)$-module M if M is a direct summand of the induced module L^G for some indecomposable $K(Q)$-module L, and Q is a minimal such subgroup. In this situation L is called a source for M. Whenever M has vertex Q and H is a subgroup of G containing Q, then M is a direct summand of $(M_H)^G$. In this situation the vertex and source of M are determined up to conjugates (Green [6]). Any further terminology and notation can be found in Curtis and Reiner [3].

From now on we assume G has a cyclic p-Sylow subgroup. Let Q be a p-subgroup of G and N its normalizer. The first proposition was proved by Green in [5] but the additional hypothesis of this section enables us to give a very short proof.

PROPOSITION. *There is a one-to-one correspondence between the set $\{M\}$ of indecomposable $K(G)$-modules with vertex Q and the set $\{W\}$ of indecomposable $K(N)$-modules with vertex Q. If M corresponds to W then M is a direct summand of W^G and W is a direct summand of M_N.*

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Proof. Let $M_N = W_1 \oplus \cdots \oplus W_t$ be a decomposition of M_N by indecomposable $K(N)$-modules. Because M is a component of $(M_N)^G$, M must be a component of W^G for W one of the W_i. We show this W is unique. The Mackey formula (44.2) of [3] gives

$$(W^G)_Q = \sum_D W(D).$$

where the sum is taken over the distinct double cosets $D = QzN$ for z in G, and $W(D) = \{W_{Q \cap zNz^{-1}}\}^Q$. We can have $Q = Q \cap zNz^{-1}$ if and only if $z^{-1}Qz \subseteq N$. Because the p-Sylow subgroup of G is cyclic, this can happen if and only if $z^{-1}Qz = Q$; that is $z \in N$. Hence there is exactly one D for which $W(D) = W_Q$ and for all others, $W(D)$ is a sum of modules with vertex properly contained in Q. But now M is a direct summand of W^G so M_Q is a direct summand of $(W^G)_Q$. Moreover W_Q is a direct summand of M_Q. Hence all other summands of M_Q have vertex properly contained in Q. In particular all $K(N)$-summands of M_N other than W have vertex $\neq Q$. Hence W is unique.

Now we want to count the number of nonisomorphic indecomposable $K(G)$-modules. We shall proceed in the following way: For a p-subgroup Q we first count the number of indecomposable $K(G)$-modules with vertex Q. By Proposition 1 it is sufficient to count the number of indecomposable $K(N(Q))$-modules with vertex Q. To accomplish this we count the number of nonisomorphic direct summands of the induced module L^N when L is an indecomposable $K(Q)$-module with vertex Q. The last step is to count the number of choices for L.

Let Q be a p-subgroup with generator $y \neq 1$. The ideals $K(Q)(y-1)^i$, $i = 0, \ldots, |Q|-1$, give a full set of isomorphism types of indecomposable $K(Q)$-modules. If L is such an ideal, the induced module L^N can be identified with the left ideal $K(N)L$. For any $x \in N$ we have $x^{-1}Lx = L$ because $x^{-1}K(Q)x = K(Q)$ and the ideals of $K(Q)$ are linearly ordered. Thus $K(N)L$ is a two-sided ideal. Let $\{e_i\}$ be a full set of mutually orthogonal primitive idempotents of $K(N)$. Since $K(N)L$ is an ideal we have a decomposition

$$K(N)L = \sum \oplus K(N)Le_i.$$

Each $K(N)Le_i$ is indecomposable since $K(N)e_i$ has a unique minimal submodule. We also have $K(N)Le_i \neq (0)$ for the following reason. By (61.3) of [3] the dimension of a left ideal plus the dimension of its right annihilator is the dimension of the algebra. If L_0 is the annihilator of L in $K(Q)$ then

$$(K(N)L : K) + (L_0K(N) : K) = [N : Q]\{(L : K) + (L_0 : K)\} = |N|.$$

Hence $L_0K(N) =$ right annihilator of $K(N)L$. Since $Q \triangle N$ the ideal $L_0K(N)$ is in the radical of $K(N)$. In particular $L_0K(N)$ contains no idempotent e_i.

Finally $K(N)e_i \cong K(N)e_j$ implies $K(N)Le_i \cong K(N)Le_j$ because $LK(N)=K(N)L$; $K(N)Le_i \cong K(N)Le_j$ implies $K(N)Le_i$ and $K(N)Le_j$ have isomorphic minimal submodules and hence $K(N)e_i \cong K(N)e_j$ by (58.12) of [3]. Hence the number of distinct indecomposable summands of $K(N)L$ is the number of distinct principal indecomposable $K(N)$-modules. By (83.5) of [3] this number is the number of p-regular conjugate classes of N. We denote this number by $n_p(N)$.

Since Q is a normal subgroup of N, the Mackey formula gives $(L^N)_Q \cong [N:Q]\cdot L$. This shows that when L has vertex Q so does every indecomposable direct summand of L^N.

In case $Q=\langle 1\rangle$ there is one choice for L and there are $n_p(G)$ indecomposable modules with vertex Q.

For $Q=\langle y\rangle \neq \langle 1\rangle$ the ideal $K(Q)(y-1)^i$ has vertex Q if and only if it is not induced from a proper subgroup of Q. This holds if and only if p does not divide i. Hence there are $|Q|(1-1/p)$ ideals with vertex Q.

We summarize these remarks in the following result.

THEOREM. *Let G be a finite group with a cyclic p-Sylow subgroup P, and let K be an algebraically closed field of characteristic p. The number of nonisomorphic indecomposable $K(G)$-modules is*

$$i(G) = n_p(G)+\sum |Q|(1-1/p)n_p(N_G(Q))$$

where the sum is taken over all nonidentity subgroups Q of P and where $n_p(X)$ denotes the number of p-regular conjugate classes in the group X.

This can sometimes be useful when trying to compute the Cartan matrix (c_{ij}) for $K(G)$, (83.8) of [3]. If $\{U_i\}$ is a full set of nonisomorphic principal indecomposable modules for $K(G)$ then the composition length of U_i is $\sum_j c_{ij}$. Since every submodule of U_i is indecomposable there are at least $\sum_{i,j} c_{ij}$ indecomposable $K(G)$-modules. This number will be the total number of indecomposable modules if and only if every indecomposable module has a unique simple submodule. By a theorem of Nakayama [8] this happens if and only if $K(G)$ is a generalized uniserial algebra. That is each U_i has a unique composition series. So we have the following:

COROLLARY. *Let G, K, and $i(G)$ be as in the theorem, and let $C=(c_{ij})$ be the Cartan matrix for $K(G)$. Then $\sum_{i,j} c_{ij} \leqq i(G)$ with equality if and only if $K(G)$ is a generalized uniserial algebra.*

3. **Examples.** Let $G=LF(2,p)$ for p an odd prime. The order of G is $\frac{1}{2}(p-1)p(p+1)$ and so the p-Sylow subgroup is cyclic of prime order p. Let K be an algebraically closed field of characteristic p. We shall describe all the indecomposable representations of G over K. The Cartan matrix for $K(G)$ is given in Brauer

and Nesbitt [1]. There are $r+1=\frac{1}{2}(p+1)$ principal indecomposable modules $U_0, U_1, \ldots, U_r$. Let $F_0, F_1, \ldots, F_r$ be the corresponding irreducible modules. The composition factors of the U_i are given as follows:

$$\begin{aligned} U_0 &= F_0 & U_i &\sim 2F_i+F_{i-1}+F_{i+1} \qquad 1 < i < r,\\ U_1 &\sim 2F_1+F_2 & U_r &\sim 3F_r+F_{r-1}. \end{aligned}$$

To determine the indecomposable modules we must first determine the arrangement of the composition factors of the U_i. The following notation will be convenient. If $J=$radical $K(G)$ and M is a $K(G)$-module we write

$$M \sim (i_1, \ldots, i_s | j_1, \ldots, j_t | \ldots)$$

to indicate that

$$M/JM \cong F_{i_1} \oplus \cdots \oplus F_{i_s}, \quad JM/J^2M \cong F_{j_1} \oplus \cdots \oplus F_{j_t}, \ \ldots .$$

From the properties of the injective hull of a module (§57 [3]) it follows that a module M with a unique minimal submodule isomorphic to F_i must be a submodule of U_i. Using this we determine that

$$\begin{aligned} U_1 &\sim (1|2|1),\\ U_i &\sim (i|i-1, i+1|i), \qquad 1 < i < r.\\ U_r &\sim (r|r-1, r|r), \end{aligned}$$

Let V_i denote the unique maximal submodule of U_i. We arrange the V_i according to the following scheme:

$$(*) \qquad \begin{aligned} &V_1, V_3, \ldots, V_{r-2}, V_r, V_{r-1}, V_{r-3}, \ldots, V_2 \qquad r \text{ odd},\\ &V_1, V_3, \ldots, V_{r-1}, V_r, V_{r-2}, V_{r-4}, \ldots, V_2 \qquad r \text{ even}. \end{aligned}$$

If V_a immediately precedes V_b in (*) then there is a unique index d such that V_a and V_b have a common homomorphic image isomorphic to F_d. Let γ_a, λ_b be homomorphisms such that $\gamma_a(V_a)=F_d=\lambda_b(V_b)$. Then γ_a, λ_b are defined for $a\neq 2$ and $b\neq 1$.

Now let $V_a, \ldots, V_k$ be a collection of consecutive modules in the ordering (*) containing at least two V's. Let M be the submodule of the direct sum $V_a \oplus \cdots \oplus V_k$ consisting of all tuples $(v_a, \ldots. v_k)$ such that $\gamma_s(v_s)=\lambda_t(v_t)$ for all pairs (s, t) for which V_s immediately precedes V_t. We may repeat this construction to obtain M' by replacing V_k with the smallest submodule of V_k which is mapped onto $\lambda_k(V_k)$ by λ_k. If $V_a\neq V_1$ we may also replace V_a with the smallest submodule of V_a that maps onto $\gamma_a(V_a)$ under γ_a to obtain M''. Finally if $V_a\neq V_1$ we can replace both V_a and V_k by appropriate submodules to get M'''. We shall prove in the appendix that M, M', M'', M''' are actually indecomposable. Observe that the socle of any of the modules here is the same as the socle of $V_a \oplus \cdots \oplus V_k$—namely $F_a \oplus \cdots \oplus F_k$. This is immediate because $\gamma_s(F_s)=0=\lambda_t(F_t)$. Hence the modules constructed in this way have different socles or else the same socles but different composition factors. So they are all different. Taking into account the $4+5(r-1)$ indecomposable submodules of the U_i we have accounted for $\frac{1}{2}(p^2-p+2)$ indecomposable $K(G)$-modules. The formula of §2 assures us there are no others. For a discussion of the subgroups of G see [2, Chapter 20].

It is interesting to note that the modules constructed are independent of the maps γ_a, λ_b.

Appendix. The purpose of this appendix is to discuss the subdirect sum of two modules over an arbitrary ring and give a test to determine when such a module is decomposable. In general subdirect sums of modules are very complicated to deal with. This case will be no exception. However when one of the modules in question is small enough, say with a unique minimal submodule, the test becomes manageable.

Let A be an arbitrary ring and M_1, M_2 two left A-modules. Suppose there are homomorphisms φ_1, φ_2 of M_1, M_2 onto a common image F. We construct a submodule M of $M_1 \oplus M_2$ by taking all pairs (m_1, m_2) for which $\varphi_1(m_1)=\varphi_2(m_2)$. Let π_i be the projection of M into M_i. We have $\pi_i(M)=M_i$, $i=1, 2$.

Conversely suppose M is a submodule of $M_1 \oplus M_2$ with $\pi_i(M)=M_i$. Let π_i be restricted to M and set

$$F = \pi_1(M)/\pi_1(\ker \pi_2).$$

Define φ_1 on M_1 to be the natural map $M_1 \to F$. Let φ_2 be defined on M_2 by setting $\varphi_2(m_2)=\varphi_1(m_1)$ provided the pair (m_1, m_2) is in M. By choice of F, φ_2 is well-defined and M is the submodule of $M_1 \oplus M_2$ constructed as above from φ_1 and φ_2. We use the notation $M=\{M_i; \varphi_i\}$ for this module.

Now suppose $M=\{M_i; \varphi_i\}$ is decomposable, say $M=L_1 \oplus L_2$ with $L_i \neq (0)$. Then $L_i \subset M_1 \oplus M_2$ so L_i is a subdirect sum, not necessarily of M_1, M_2 but of $\pi_j(L_i)=M_{ji}$. There are maps λ_{ji} such that $\lambda_{ji}(L_i)=E_i$ for some E_i and

$$L_j = \{M_{ij}; \lambda_{ij}\} \qquad 1 \leqq i, j \leqq 2.$$

Of course there are restrictions on the λ_{ij} imposed by the fact that $L_i \subset M$. The precise relation is given in the following proposition.

PROPOSITION. *Let $M=\{M_i; \varphi_i\}$ $i=1, 2$. Then M is decomposable if and only if there exist modules M_{ij}, E_j and maps λ_{ij}, k_{ij}, γ_j for $i, j=1, 2$ such that the following statements are true.*

I. *Every square in the diagram below is commutative.*

II. $\lambda_{ij}(M_{ij})=E_j$.

III. *For each* j, M_{1j} *and* M_{2j} *are not both the zero module.*

IV. *For each pair* (m_1, m_2) *in* M *there are uniquely determined pairs* (m_{1j}, m_{2j}) *in* $M_{1j} \oplus M_{2j}$ *such that* $\lambda_{1j}(m_{1j}) = \lambda_{2j}(m_{2j})$ *and* $k_{i1}(m_{i1}) + k_{i2}(m_{i2}) = m_i$ *for* $i, j = 1, 2$.

$$\begin{tikzcd}
M_{11} \arrow[r, "k_{11}"] \arrow[d, "\lambda_{11}"] & M_1 \arrow[d, "\varphi_1"] & M_{12} \arrow[l, "k_{12}"'] \arrow[d, "\lambda_{12}"] \\
E_1 \arrow[r, "\gamma_1"] & F & E_2 \arrow[l, "\gamma_2"'] \\
M_{21} \arrow[r, "k_{21}"] \arrow[u, "\lambda_{21}"] & M_2 \arrow[u, "\varphi_2"] & M_{22} \arrow[l, "k_{22}"'] \arrow[u, "\lambda_{22}"]
\end{tikzcd}$$

DIAGRAM

Proof. Suppose I–IV hold. Let $L_i' = \{M_{ji}; \lambda_{ji}\}$ and $L_i = (k_{1i} \oplus k_{2i})L_i'$. By I, $L_i \subseteq M$ and $L_i \neq (0)$ by III. By IV we have $L_1 \oplus L_2 \cong M$.

Conversely suppose $M = L_1 \oplus L_2$ with each $L_i \neq (0)$. Construct M_{ij}, λ_{ij} as in the paragraph preceding the statement of the proposition. Let k_{ij} be inclusions and define γ_i so that the upper square commutes. That is for $m_{1i} \in M_{1i}$ let $\gamma_i(\lambda_{1i}(m_{1i})) = \varphi_1(k_{1i}(m_{1i}))$. Suppose $\lambda_{11}(m_{11}) = 0$. Then there is an element $(m_{11}, 0)$ in L_1 and hence in M. Thus $\varphi_1(m_{11}) = 0$ so γ_1 is well defined. Similarly γ_2 is well defined. We now check that the lower squares commute. Pick $m_{21} \in M_{21}$. There is an element (m_{11}, m_{21}) in L_1 for which $\lambda_{11}(m_{11}) = \lambda_{21}(m_{21})$. Thus

$$\gamma_1 \circ \lambda_{21}(m_{21}) = \gamma_1 \circ \lambda_{11}(m_{11}) = \varphi_1 \circ k_{11}(m_{11}) = \varphi_1(m_{11}).$$

The element (m_{11}, m_{21}) must also belong to M so $\varphi_1(m_{11}) = \varphi_2(m_{21})$. Thus

$$\gamma_2 \circ \lambda_{21}(m_{21}) = \varphi_2(m_{21})$$

as required. Use a similar argument for the lower right hand square. This proves I. Statement II follows from the construction. The fact that $L_i \neq (0)$ implies III and statement IV follows because the sum $L_1 \oplus L_2$ is a direct sum.

It may be difficult to use this proposition because there is no indication how one should choose the M_{ij} having been given only $\{M_i; \varphi_i\}$. We list three facts which may help to determine the M_{ij} in certain cases. Again consider the k_{ij} as inclusions.

$$M_{i1} + M_{i2} = M_i \qquad \text{for } i = 1, 2. \tag{1}$$

$$\ker \lambda_{i1} \cap \ker \lambda_{i2} = (0) \qquad \text{for } i = 1, 2. \tag{2}$$

$$\gamma_1(E_1) + \gamma_2(E_2) = F. \tag{3}$$

Statement (2) follows because a nonzero element in the intersection can be used to violate the uniqueness in IV for the element $(m_1, m_2) = (0, 0)$.

We now apply this proposition to show the modules constructed in the previous section are indecomposable. If any of the modules can be decomposed then there is a smallest one that can be decomposed. Let $V_a, \ldots, V_e, V_k$ be the shortest sequence in the ordering (*) for which one of the four (or possibly two if $V_a = V_1$) modules M, M', M'', M''' is decomposable. We suppose M is the one, the treatment of the other cases being almost exactly the same.

In order to apply the proposition we first describe M as a subdirect sum of two modules, M_1 and M_2. For M_1 take V_k and for φ_1 take λ_k. For M_2 take the module constructed from the sequence $V_a, \ldots, V_e$ (or take $M_2 = V_a$ if $V_e = V_a$) and for φ_2 take the map $\gamma_e \circ \pi_e$ where π_e is the projection onto V_e. Then we have $M = \{M_i; \varphi_i\}$. We suppose $M = L_1 \oplus L_2$ with $L_i \neq (0)$ and so we have a collection of modules and maps as given in the proposition and the diagram such that I–IV are valid.

M_1 has a unique minimal submodule so nonzero submodules have a nonzero intersection. Thus by statement (2) one of the maps λ_{11}, λ_{12} is one-to-one. Suppose it is λ_{12}. Then E_2 is a submodule of M_1 and also a homomorphic image of a submodule of M_2. By inspecting the composition factors of M_2 we see there cannot be a submodule E_2 of M_1 which is a homomorphic image of a submodule of M_2 and which also maps onto F. Thus $\gamma_2(E_2) = (0)$ and $M_{22} \subseteq \ker \varphi_2$.

If we also take into account the arrangement of the composition factors of M_2 we see that E_2 must be F_k (the minimal submodule of M_1) or (0), for no other submodule is a homomorphic image of a submodule of $\ker \varphi_2$. Since λ_{12} is one-to-one we have $M_{12} = F_k$ or $M_{12} = (0)$. In either case statement (1) above implies $M_{11} = M_1$. Since no submodule of M_2 can map onto M_1 we see λ_{11} is not one-to-one. In particular

$$F_k \subseteq \ker \lambda_{11}. \tag{4}$$

Now take any $m_2 \in \ker \varphi_2$. By IV we have

$$(0, m_2) = (m_{11}, m_{21}) + (-m_{11}, m_{22}).$$

The element $-m_{11}$ is in $M_{12} \subseteq F_k$ and so by (4) $\lambda_{11}(m_{11}) = (0)$. By the condition in IV, $\lambda_{21}(m_{21}) = 0$. This proves

$$\ker \varphi_2 = \ker \lambda_{21} + M_{22}. \tag{5}$$

Now choose $z \in \ker \lambda_{21} \cap M_{22}$. There is some $m_{12} \in F_k$ such that $(m_{12}, z) \in L_2$. By (4), $\lambda_{11}(m_{12}) = 0 = \lambda_{21}(z)$ so that $(m_{12}, z) \in L_1$. Since $L_1 \oplus L_2$ is a direct sum we have $z = 0$. In particular (5) is a direct sum decomposition of $\ker \varphi_2$.

However $\ker \varphi_2$ is a module of the type M' constructed from the sequence $V_a, \ldots, V_e$. By the choice of M we must have $\ker \varphi_2$ indecomposable. Hence either $\ker \lambda_{21} = (0)$ or $M_{22} = (0)$.

Consider the first choice. If λ_{21} is one-to-one, then M_{21} is a homomorphic image of M_1 $(= M_{11})$ with kernel containing F_k. Thus M_{21} is completely reducible. If $M_{21} = U \oplus (M_{21} \cap M_{22})$ then by statement (1) we have $M_2 = U \oplus M_{22}$. Again our choice of M implies M_2 is indecomposable. If $U = 0$ then $M_2 = M_{22} \subseteq \ker \varphi_2$.

This is impossible. If $M_{22}=(0)$ then $E_2=(0)$ and $M_{12}=(0)$ since λ_{12} is one-to-one. But this violates III in the proposition. Thus ker $\lambda_{21}\neq(0)$ so $M_{22}=(0)$. But we have just seen $M_{22}=(0)$ gives a contradiction. Hence the assumption that M is decomposable is not valid.

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